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AKTS.
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Compactly generated t-structures on $D(R)$ (after Hrbek).

1. t-structures.

2. Give classification of c.g. t-structures on $D(R)$.

3. Relation to my work.

$\boxed{1}$ $\mathcal{D}$ a Δed cat. A <u>t-structure</u> is given by $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0} \subseteq \mathcal{D}$ s.t.
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- $\mathcal{D}_{\geq 0}[1] \subseteq \mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 0}[-1]$, $\mathcal{D}_{\leq 0}$,
- $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$, $X \in \mathcal{D}_{\geq 0}$, $Y \in \mathcal{D}_{\leq -1} := \mathcal{D}_{\leq 0}[-1]$,
- For any $X \in \mathcal{D}$, there is a fib. seq.

$$\begin{array}{ccccc} X_{\geq 0} & \longrightarrow & X & \longrightarrow & X_{\leq -1}. \\ \cap & & & & \cap \\ \mathcal{D}_{\geq 0} & & & & \mathcal{D}_{\leq 0}[-1] \end{array}$$

<u>Notation</u>. $\mathcal{D}_{\geq n} = \mathcal{D}_{\geq 0}[n]$, $\mathcal{D}_{\leq n} = \mathcal{D}_{\leq 0}[n]$.

<u>Examples</u>. (a) Trivial: $\mathcal{D}_{\geq 0} = \mathcal{D}$, $\mathcal{D}_{\leq 0} = 0$ or vice versa.

(b) "Standard" t-structure on $D(R)$.

$$\mathcal{D}_{\geq 0} = \{ X \in D(R) : H_i(X) = 0, i < 0 \},$$
$$\mathcal{D}_{\leq 0} = \{ X \in D(R) : H_i(X) = 0, i > 0 \}.$$

(c) Torsion t-structure. $D(R)_{\geq 0} = \{ X \in D(R), H_i(X) \text{ p-torsion for all } i \}$ $= e \}$.
Pick $p \in \mathrm{Spec}(R)$
$D(R)_{\leq 0} = \{ X \in D(R), H_i(X) \text{ is p-torsion free for all } i \}$.

(d) Sp, $Sp_{\geq 0} = Sp^{cn}$.

Def. A t-structure is *generated* by $S \subseteq D$ if

$D_{\leq 0}[-1] = S^{\perp}$ and $D_{\geq 0} = {}^{\perp}(S^{\perp})$.

It is *compactly gen* if S consists of compact objects.

$\boxed{2}$ R commutative.

Thomason subset $A \subseteq \operatorname{Spec} R$: $A = \bigcup_{\text{f.g. ideal } I} V(I)$.

Thm (Hrbek). There is a 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Thomason filtrations} \\ \cdots \supseteq A_{i+1} \supseteq A_i \supseteq A_{i-1} \supseteq \cdots \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Compactly gen. t-structures} \\ \text{on } D(R) \end{array} \right\}.$$

$$\{A_*\} \longmapsto D(R)_{\geq 0} = \{X \in D(R) : H_i(X) \text{ is supp. inside } A_i \text{ for each } i\}.$$

$$\begin{array}{c} A_i = \bigcup V(I) \\ R/I[i] \in D(R)_{\geq 0} \\ I \text{ f.g.} \end{array} \longleftarrow D(R)_{\geq 0}.$$

Compactly gen. by

$S = \{ K(I)[i] : \forall I \text{ s.t. } V(I) \subseteq A_i,\ I \text{ f.g.} \}$.

Koszul cplx.

Exs. (a) $\mathcal{D}(R)_{\geq 0} = \mathcal{D}(R) = \{\operatorname{Spec} R\}$ for all n,

$\mathcal{D}(R)_{\leq 0} = \mathcal{D}(R) = \{\emptyset\}$.

(b) Standard. $\cdots \subseteq \emptyset \subseteq \emptyset \subseteq \emptyset \subseteq \underset{0}{\operatorname{Spec} R} \subseteq \operatorname{Spec} R \subseteq \cdots$

(c) Torsion $\Big\{ \underset{\underset{A_i}{\shortparallel}}{\overline{\{p\}}} \Big\}$.

Where does this come from?

On Mod_R, a torsion pair is ~~[illegible]~~

$T, F \subseteq \mathrm{Mod}_R$ s.t. $T^{\perp} = F$, ${}^{\perp}F = T$.

$$\left\{ \begin{matrix} \text{Thomason subsets} \\ \text{of } \operatorname{Spec} R \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Hereditary torsion pairs} \\ \text{of f.t. in } \mathrm{Mod}_R \end{matrix} \right\}.$$

$\boxed{3}$ Thm (AGH). E a small stable ∞-cats w/ bounded t-structure, $K_{-1}(E) = 0$ and if $E^{\heartsuit}$ is noetherian, $K_{-n}(E) = 0$ for all $n \geq 1$.

Goal: classify bounded t-structures on $\mathrm{Perf}(R)$, R commutative.

Singular case: none expected.

- The standard structure on $\mathcal{D}(R)$ does not restrict to Perf if R is singular noetherian.
- None for Artinian singular.

Regular case: Definitely some but def. not all restrict to $\mathrm{Perf}(R) \simeq D^b(R)$.

- Standard ~~[illegible]~~ restricts.
- Some (at least) of the torsion t-structures don't restrict.
- Could help with positive K-theory results.

Prop. E stable ∞-cat with t-structure $\leadsto$ c.g. t-structure $\mathrm{Ind}(E_{\geq 0})$ on $\mathrm{Ind}(E)$.

- $\mathrm{Ind}(\mathrm{Perf}(R)) \simeq D(R)$. So, t-structures on $\mathrm{Perf}(R)$ give c.g. t-structures on $D(R)$.